*Congruences Associated with a One-Parameter Family of Curves.**

By Ralph Dennison Beetle.

Introduction.

With a one-parameter family of curves may be associated a number of rectilinear congruences. The congruence formed by the tangents to the curves has been very extensively studied, and its properties have been found to be intimately connected with the nature of the surface on which the curves lie and with the relation of the curves to this surface. In this paper, we consider some of the other rectilinear congruences and also certain congruences of circles associated with the curves. Particular attention is given to the rectilinear congruences formed by the principal normals and binormals, and the congruence of circles formed by the osculating circles. The discussion in this preliminary paper is restricted to rather elementary properties of the congruences.

The paper is divided into four parts. In the first part, we state those general formulas relating to one-parameter families of curves which are necessary in the later portions of the paper. It is found advantageous to use the method recently suggested by Eisenhart.† In this method, the moving triedral formed by the tangent, the principal normal and the binormal serves as a frame of reference, and the treatment of problems relating to these lines is thereby essentially simplified.

The second part of the paper is devoted to rectilinear congruences. As is well known, it is a characteristic property of a system of geodesics that the congruence of tangents is normal, and of a system of asymptotic lines that the surface on which they lie is the middle surface of the congruence of tangents. When the congruence of principal normals or binormals is normal, or has the surface for its middle surface, the resulting geometric property of the curves is not so tangible, but a consideration of these properties of the congruences leads to a number of general theorems of interest.

---

* Presented to the American Mathematical Society, September 8, 1914.

† L. P. Eisenhart, "One-Parameter Families of Curves," American Journal of Mathematics, Vol. XXXVII (1915), p. 179.

Of the other rectilinear congruences discussed in the second part, we mention here only the congruence of polar lines. By considering this congruence, we find the following characterization of a system of lines of curvature. In order that a one-parameter family of curves be a system of lines of curvature of the surface on which the curves lie, it is necessary and sufficient that the point of meeting of the normal to the surface and the corresponding polar line be a focal point of the congruence of polar lines.

In the third part of the paper, we determine under what conditions the congruence of osculating circles is a cyclic system. We find that it is necessary and sufficient that the curves in question be lines of curvature of constant geodesic curvature. Hence, they are either plane geodesics or else spherical curves which lie on spheres orthogonal to the surface formed by the curves. In this part of the paper, we also discuss cyclic systems in which the circles lie on a single infinity of planes or of spheres.

In connection with the notions and results of the second part, it is interesting to consider the surfaces characterized by the fact that the asymptotic lines in one or both systems are geodesic parallels. These surfaces have apparently not previously been discussed. The fourth part of the paper deals with these surfaces. The determination of all such surfaces requires the solution of a rather complicated partial differential equation of the fourth order. A number of characteristic properties of these surfaces are found.

I. ONE-PARAMETER FAMILIES OF CURVES.

§ 1. *Equations of Condition.*

In the paper mentioned above, Eisenhart shows that, if p, q, r, t, ρ and τ satisfy the three conditions

$$\left.\begin{aligned} \frac{\partial q}{\partial u}-\frac{\partial p}{\partial v}-\frac{pr}{\rho}=0, \\ \frac{\partial A_2}{\partial u}-\frac{\partial}{\partial v}\left(\frac{p}{\rho}\right)+\frac{p}{\tau}A_3=0, \\ \frac{\partial L_3}{\partial u}+\frac{\partial}{\partial v}\left(\frac{p}{\tau}\right)+\frac{p}{\rho}A_3=0, \end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned} A_2&=\frac{1}{p}\frac{\partial r}{\partial u}+\frac{q}{\rho}+\frac{t}{\tau}, \\ A_3&=\frac{1}{p}\frac{\partial t}{\partial u}-\frac{r}{\tau}, \\ L_3&=\frac{\rho}{p}\frac{\partial A_3}{\partial u}-\frac{\rho}{\tau}A_2, \end{aligned}\right\} \tag{2}$$

then the system of equations, consisting of the six equations

$$\left.\begin{aligned} \frac{\partial\alpha}{\partial u} &= \frac{p}{\rho}\,l, & \frac{\partial\alpha}{\partial v} &= \quad A_2\,l + A_3\,\lambda, \\ \frac{\partial l}{\partial u} &= -\frac{p}{\rho}\,\alpha - \frac{p}{\tau}\,\lambda, & \frac{\partial l}{\partial v} &= -A_2\,\alpha + L_3\,\lambda, \\ \frac{\partial\lambda}{\partial u} &= \frac{p}{\tau}\,l, & \frac{\partial\lambda}{\partial v} &= -A_3\,\alpha - L_3\,l, \end{aligned}\right\} \tag{3}$$

together with those obtained by replacing α, l, λ by β, m, μ, and by γ, n, ν, is completely integrable, and admits solutions such that the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$$

is orthogonal and positive. Under these conditions, the equations

$$\frac{\partial x}{\partial u} = p\alpha, \quad \frac{\partial x}{\partial v} = q\alpha + rl + t\lambda, \tag{4}$$

and the analogous ones for y and z, are consistent, and the locus of the point $P(x, y, z)$ is a surface S.

The surface S may be regarded as the locus of the one-parameter family of curves C obtained by assigning arbitrary constant values to v. Then α, β, γ; l, m, n; λ, μ, ν are, respectively, the direction-cosines of the tangent, principal normal and binormal of the curve C through the corresponding point P; ρ and τ are the radii of first and second curvature of the curve C.

Conversely, if $x, \alpha, l, \lambda, \rho, \tau$, etc., have the significance just indicated, and the curves C are not minimal or straight lines, the p, q, r and t defined by (4) and the analogous equations in y and z satisfy the equations (1).

§ 2. *Fundamental Quantities for the Surface S. Special Parametric Systems.*

If the linear element of the surface S is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

it follows from (4) that

$$E = p^2, \quad F = pq, \quad G = q^2 + r^2 + t^2, \tag{5}$$

whence

$$H = \sqrt{EG - F^2} = p\sqrt{r^2 + t^2}. \tag{6}$$

If X, Y, Z are the direction-cosines of the normal to the surface,

$$X = \frac{r\lambda - tl}{\sqrt{r^2 + t^2}} = \lambda \sin\omega + l\cos\omega, \tag{7}$$

where ω is the angle which the normal to the surface makes with the principal

normal of the curve C at the corresponding point. We note for reference that

$$\sin\omega=\frac{r}{\sqrt{r^2+t^2}},\quad \cos\omega=-\frac{t}{\sqrt{r^2+t^2}},\quad \tan\omega=-\frac{r}{t}. \tag{8}$$

If the second fundamental quadratic form* of the surface S is

$$-\Sigma dx dX=D du^2+2D' du dv+D'' dv^2,$$

it is found that

$$\left.\begin{aligned} D\ &=-\frac{p^2 t}{\rho\sqrt{r^2+t^2}}\\ D'\ &=\frac{p(rA_3-tA_2)}{\sqrt{r^2+t^2}}=\frac{pq}{\rho}\cos\omega+\left(\frac{\partial\omega}{\partial u}-\frac{p}{\tau}\right)\sqrt{r^2+t^2},\\ D''&=\frac{q(rA_3-tA_2)}{\sqrt{r^2+t^2}}+\left(L_3+\frac{\partial\omega}{\partial v}\right)\sqrt{r^2+t^2}.\end{aligned}\right\} \tag{9}$$

Since we have excluded the case in which the curves C are minimal, we may always assume that $p\neq 0$. If $p=1$, the parameter u is the arc of the curve C. Whenever p is a function of u alone, we may take $p=1$, since this result can be secured by a change of parameters which preserves the parametric curves.

We have excluded also the case in which the curves C are rulings. Then the necessary and sufficient condition that they be geodesics is that $r=0$. In order that they be asymptotic lines, it is necessary and sufficient that $t=0$.

If $q=0$, the parametric system is orthogonal, and conversely. The parametric system is conjugate if

$$rA_3-tA_2=0; \tag{10}$$

it consists of the lines of curvature if

$$q=0,\quad \frac{\partial\omega}{\partial u}-\frac{p}{\tau}=0, \tag{11}$$

in view of (5) and (9). In consequence of a well known property of the lines of curvature, the second of the conditions (11) is necessary and sufficient that the curves C be lines of curvature.†

§ 3. *The Quantities* ω, r *and* t.

From its definition, it is evident that the value of ω at a given point depends only on the curves C. It is readily proved that the values of r and t are independent of the choice of the curves $u=$const., but do depend on the particular distribution of the parameter v in the system of curves C. If, how-

---

* Cf. Eisenhart, "Differential Geometry," p. 114 (Ginn and Co., Boston, 1909). Hereafter, a reference to this book will be given in the form Eisenhart, p. 114.

† Eisenhart, p. 139.

ever, the parameter v is fixed, the values of ω, r and t are all uniquely determined. Since the system of curves C is arbitrary, except that we have excluded minimal and straight lines, it follows that we may associate with any system of curves defined by an equation of the form $\psi(u, v) = \text{const.}$ three functions ω_ψ, r_ψ and t_ψ.

Moreover, since any curve on the surface S, except a curve $u=\text{const.}$, can be defined by an equation of the form $v-\phi(u)=0$, and hence be regarded as belonging to the family given by $v-\phi(u)=\text{const.}$, we may define the three functions for any curve of the surface. In this case, we shall denote them by ω_ϕ, r_ϕ and t_ϕ. We proceed to find expressions for them.

From the well known formulas for normal curvature and geodesic curvature* it follows that

$$\tan\omega_\phi = \frac{E+2F\dfrac{\partial\phi}{\partial u}+G\left(\dfrac{\partial\phi}{\partial u}\right)^2}{H\left[D+2D'\dfrac{\partial\phi}{\partial u}+D''\left(\dfrac{\partial\phi}{\partial u}\right)^2\right]}\left\{\frac{\partial}{\partial u}\left[\frac{F+G\dfrac{\partial\phi}{\partial u}}{\sqrt{E+2F\dfrac{\partial\phi}{\partial u}+G\left(\dfrac{\partial\phi}{\partial u}\right)^2}}\right] - \frac{\partial}{\partial v}\left[\frac{E+F\dfrac{\partial\phi}{\partial u}}{\sqrt{E+2F\dfrac{\partial\phi}{\partial u}+G\left(\dfrac{\partial\phi}{\partial u}\right)^2}}\right]\right\} \tag{12}$$

In view of (5), (6) and (8),

$$r = \frac{H}{\sqrt{E}}\sin\omega \tag{13}$$

and

$$t = -\frac{H}{\sqrt{E}}\cos\omega. \tag{14}$$

Therefore we conclude that†

$$r_\phi = \frac{H\sin\omega_\phi}{\sqrt{E+2F\dfrac{\partial\phi}{\partial u}+G\left(\dfrac{\partial\phi}{\partial u}\right)^2}}, \tag{15}$$

and

$$t_\phi = -\frac{H\cos\omega_\phi}{\sqrt{E+2F\dfrac{\partial\phi}{\partial u}+G\left(\dfrac{\partial\phi}{\partial u}\right)^2}}. \tag{16}$$

§ 4. *The Curves $u=const.$*

If α_1, l_1, λ_1, p_1, q_1, r_1, t_1 ρ_1, τ_1, etc., are similarly defined for the curves $u=\text{const.}$, the following relations are easily deduced:

---

* Eisenhart, pp. 117, 132, 136. † Eisenhart, p. 74.

$$\alpha_1 = \frac{q\alpha + rl + t\lambda}{\sqrt{q^2+r^2+t^2}}; \tag{17}$$

$$l_1 = \frac{a_1\alpha + a_2 l + a_3\lambda}{\sqrt{a_1^2+a_2^2+a_3^2}}; \tag{18}$$

$$\lambda_1 = \frac{(ra_3 - ta_2)\alpha + (ta_1 - qa_3)l + (qa_2 - ra_1)\lambda}{\sqrt{q^2+r^2+t^2}\sqrt{a_1^2+a_2^2+a_3^2}}, \tag{19}$$

where

$$\left.\begin{aligned} a_1 &= \frac{\partial}{\partial v}\left(\frac{q}{\sqrt{q^2+r^2+t^2}}\right) - \frac{rA_2+tA_3}{\sqrt{q^2+r^2+t^2}}, \\ a_2 &= \frac{\partial}{\partial v}\left(\frac{r}{\sqrt{q^2+r^2+t^2}}\right) + \frac{qA_2-tL_3}{\sqrt{q^2+r^2+t^2}}, \\ a_3 &= \frac{\partial}{\partial v}\left(\frac{t}{\sqrt{q^2+r^2+t^2}}\right) + \frac{qA_3+rL_3}{\sqrt{q^2+r^2+t^2}}. \end{aligned}\right\} \tag{20}$$

We have also

$$\left.\begin{aligned} p_1 &= \sqrt{q^2+r^2+t^2}, \\ q_1 &= \frac{pq}{\sqrt{q^2+r^2+t^2}}, \\ r_1 &= \frac{pa_1}{\sqrt{a_1^2+a_2^2+a_3^2}}, \\ t_1 &= \frac{p(ra_3-ta_2)}{\sqrt{q^2+r^2+t^2}\sqrt{a_1^2+a_2^2+a_3^2}}, \end{aligned}\right\} \tag{21}$$

and

$$\left.\begin{aligned} \frac{1}{\rho_1} &= \frac{\sqrt{a_1^2+a_2^2+a_3^2}}{\sqrt{q^2+r^2+t^2}}, \\ \frac{1}{\tau_1} &= \frac{\sqrt{(b_1^2+b_2^2+b_3^2)-(a_1^2+a_2^2+a_3^2)}}{\sqrt{q^2+r^2+t^2}}, \end{aligned}\right\} \tag{22}$$

where the quantities b_1, b_2 and b_3 are obtained by replacing q, r and t by a_1, a_2 and a_3 in the expressions (20) for a_1, a_2 and a_3.

II. RECTILINEAR CONGRUENCES.

§ 5. *Congruences* $\boldsymbol{\Gamma}$. *Notation.*

All of the rectilinear congruences studied in this paper are congruences generated by a line L whose direction, relative to the moving triedral of reference, remains invariable. Such a congruence we denote by the symbol $\boldsymbol{\Gamma}$. If $P_0(x_0, y_0, z_0)$ is any point, fixed or variable, on the moving line L, and X_0, Y_0, Z_0 are the direction-cosines of the line, we have

$$x_0 = x + \xi_1 \alpha + \xi_2 l + \xi_3 \lambda, \quad X_0 = c_1 \alpha + c_2 l + c_3 \lambda, \tag{23}$$

where ξ_1, ξ_2, ξ_3 are functions of u and v, but c_1, c_2, c_3 are constants such that $c_1^2 + c_2^2 + c_3^2 = 1$.

We write

$$\left.\begin{aligned}
&\mathcal{E}_0 = \Sigma \left(\frac{\partial X_0}{\partial u}\right)^2, \quad \mathcal{F}_0 = \Sigma \frac{\partial X_0}{\partial u}\frac{\partial X_0}{\partial v}, \quad \mathcal{G}_0 = \Sigma \left(\frac{\partial X_0}{\partial v}\right)^2, \\
&e_0 = \Sigma \frac{\partial x_0}{\partial u}\frac{\partial X_0}{\partial u}, \quad f_0 = \Sigma \frac{\partial x_0}{\partial v}\frac{\partial X_0}{\partial u}, \\
&f_0' = \Sigma \frac{\partial x_0}{\partial u}\frac{\partial X_0}{\partial v}, \quad g_0 = \Sigma \frac{\partial x_0}{\partial v}\frac{\partial X_0}{\partial v},
\end{aligned}\right\} \tag{24}$$

so that

$$\Sigma dX_0^2 = \mathcal{E}_0 du^2 + 2\mathcal{F}_0 du dv + \mathcal{G}_0 dv^2,$$

and

$$\Sigma dx_0 dX_0 = e_0 du^2 + (f_0 + f_0') du dv + g_0 dv^2.$$

Then the equation of the developable surfaces of the congruence is*

$$(\mathcal{E}_0 f_0' - e_0 \mathcal{F}_0) du^2 + (\mathcal{E}_0 g_0 - \mathcal{G}_0 e_0 + \mathcal{F}_0 f_0' - \mathcal{F}_0 f_0) du dv + (\mathcal{F}_0 g_0 - f_0 \mathcal{G}_0) dv^2 = 0. \tag{25}$$

If P_{01} and P_{02} are the focal points of the line L, their coordinates are of the form

$$x_{01} = x_0 + \rho_{01} X_0, \quad x_{02} = x_0 + \rho_{02} X_0, \tag{26}$$

where †

$$\left.\begin{aligned}
\rho_{01} + \rho_{02} &= \frac{(f_0 + f_0')\mathcal{F}_0 - g_0 \mathcal{E}_0 - e_0 \mathcal{G}_0}{\mathcal{E}_0 \mathcal{G}_0 - \mathcal{F}_0^2}, \\
\rho_{01}\rho_{02} &= \frac{e_0 g_0 - f_0 f_0'}{\mathcal{E}_0 \mathcal{G}_0 - \mathcal{F}_0^2}.
\end{aligned}\right\} \tag{27}$$

If the line L always passes through the corresponding point P, we may choose P for the point P_0. In this case, we shall speak of congruences $\boldsymbol{\Gamma}_P$. The congruences $\boldsymbol{\Gamma}_P$ formed by the tangents, principal normals and binormals, we shall denote by $\boldsymbol{\Gamma}_t$, $\boldsymbol{\Gamma}_n$ and $\boldsymbol{\Gamma}_b$. Similarly, by $\boldsymbol{\Gamma}_N$ we indicate a congruence $\boldsymbol{\Gamma}_P$ such that the line L lies in the normal plane of the curve C.

§ 6. *Normal Congruences* $\boldsymbol{\Gamma}_P$.

The condition that a congruence $\boldsymbol{\Gamma}$ be normal,‡ $f_0 = f_0'$, reduces to

$$\begin{aligned}
&\frac{\partial}{\partial u}[c_1(q - \xi_2 A_2 - \xi_3 A_3) + c_2(r + \xi_1 A_2 - \xi_3 L_3) + c_3(t + \xi_1 A_3 + \xi_2 L_3)] \\
&\quad = \frac{\partial}{\partial v}\left[c_1\left(p - \xi_2 \frac{p}{\rho}\right) + c_2\left(\xi_1 \frac{p}{\rho} + \xi_3 \frac{p}{\tau}\right) - c_3\left(\xi_2 \frac{p}{\tau}\right)\right],
\end{aligned} \tag{28}$$

* Eisenhart, p. 398. † Eisenhart, p. 399. ‡ Eisenhart, p. 393.

and becomes, in the case of congruences Γ_P,

$$c_1\left(\frac{\partial q}{\partial u}-\frac{\partial p}{\partial v}\right)+c_2\frac{\partial r}{\partial u}+c_3\frac{\partial t}{\partial u}=0,$$

or, in view of (1),

$$c_1\frac{pr}{\rho}+c_2\frac{\partial r}{\partial u}+c_3\frac{\partial t}{\partial u}=0. \tag{29}$$

The congruences Γ_t, Γ_n and Γ_b correspond to the choices $c_1=1$, $c_2=c_3=0$; $c_2=1$, $c_1=c_3=0$; $c_3=1$, $c_1=c_2=0$. We have already remarked that, if we rule out minimal and straight lines, the congruence Γ_t is normal if, and only if, $r=0$. We can now add

THEOREM 1. *The necessary and sufficient condition that the congruence of principal normals be normal is that r be constant along each curve C. The necessary and sufficient condition that the congruence of binormals be normal is that t be constant along each curve C.*

For the sake of brevity, we denote by *r-line* a curve along which r is constant, and by *t-line* a curve along which t is constant. Since geodesics are r-lines and asymptotic lines are t-lines, such lines exist on every surface. Moreover, the remarks of § 3 enable us to prove that, on every surface, there exist infinitely many r-lines and t-lines, other than geodesics and asymptotic lines. For the necessary and sufficient condition that the principal normals of a system of curves defined by

$$v-\phi(u)=\text{const.}$$

form a normal congruence is that there be a functional relation between

$$\frac{H^2\sin^2\omega_\phi}{E+2F\dfrac{\partial\phi}{\partial u}+G\left(\dfrac{\partial\phi}{\partial u}\right)^2}$$

and $v-\phi$. Equating to zero the Jacobian of these expressions and making use of (12), we find that ϕ satisfies a differential equation of the third order. A similar result is obtained in the case of t-lines.

Suppose now that the curves C are both r-lines and t-lines. Then

$$\frac{\partial r}{\partial u}=\frac{\partial t}{\partial u}=0,$$

and we can take $q=0$. In view of (5) and (8), we conclude that

$$\frac{\partial\omega}{\partial u}=\frac{\partial G}{\partial u}=0,$$

so that the curves C are geodesic parallels,* along each of which the osculating

* Eisenhart, p. 134.

plane meets the surface under constant angle. The converse theorem is easily established.

If we seek a system of curves such that all three of the congruences Γ_t, Γ_n and Γ_b are normal, it is therefore necessary and sufficient that they be geodesics whose orthogonal trajectories are geodesics. As is well known, orthogonal systems of geodesics exist only on developable surfaces; they correspond, of course, to orthogonal systems of straight lines in the plane.

From the form of the condition (29) there follows at once

THEOREM 2. *If two different lines, L_1 and L_2, generate normal congruences Γ_P, every congruence Γ_P generated by a line L in the plane of L_1 and L_2 is also normal, and is an associate* of the first two. If three non-coplanar lines generate normal congruences Γ_P, every congruence Γ_P is normal.*

§ 7. *Other Normal Congruences* Γ.

If the curves C are geodesics and the curves $u=$const. are their conjugates, we have by (10) $r=A_2=0$. If we take $c_1=c_3=0$, $c_2=1$, $\xi_2=\xi_3=0$, the condition (28) for normality becomes

$$\frac{\partial}{\partial v}\left(\xi_1\frac{p}{\rho}\right)=0, \tag{30}$$

so that we obtain a normal congruence by taking

$$\xi_1=\frac{\rho}{p}U_1,$$

where U_1 is a function of u alone. If $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ describes one of the surfaces normal to this congruence, we may write

$$\bar{x}=x+\frac{\rho}{p}U_1\alpha+\eta l,$$

where η is to be determined. Then, since we must have

$$\Sigma l\frac{\partial\bar{x}}{\partial u}=0, \quad \Sigma l\frac{\partial\bar{x}}{\partial v}=0,$$

we find that

$$\eta=-\int U_1du=-U,$$

so that

$$\bar{x}=x+\frac{\rho}{p}U'\alpha-Ul, \tag{31}$$

where the prime denotes differentiation with respect to u.

THEOREM 3. *If a geodesic system, $v=$const., and its conjugate system, $u=$const., are known on the surface S, each choice of a function of u deter-*

* Eisenhart, p. 402.

mines, without any integration, a surface $\bar{S}$ which corresponds to S with parallelism of tangent planes.

If S_1 is the other focal surface of the normal congruence of tangents to the geodesic system, the curves u=const. on S_1 are geodesics and the curves v=const. are their conjugates.* Hence, by Theorem 3, to each choice of a function of v will correspond a normal congruence in the tangent plane of S. Taking $c_1=c_2=0$, $c_3=1$, $\xi_2=\xi_3=0$, we find from (28) that the corresponding congruence is normal if

$$\xi_1=\frac{V_1-t}{A_3},$$

where V_1 is any function of v alone. The conditions $r=0$, $A_2=0$ are not needed in this case.

If $\bar{P}_1(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ describes a surface normal to this congruence, we have

$$\bar{x}_1=x+\frac{V'-t}{A_3}\alpha-V\lambda, \tag{32}$$

where $V=\int V_1 dv$.

THEOREM 4. *If any system of curves, v=const., is known on the surface S, each choice of a function of v determines, without any integration, a surface $\bar{S}_1$, the normal to which lies in the corresponding rectifying plane of the curve v=const. If the curves are geodesics, S and $\bar{S}_1$ correspond with orthogonality of tangent planes. If the curves are asymptotic lines, S and $\bar{S}_1$ correspond with parallelism of tangent planes.*

If we put $V=0$, $\bar{S}_1$ becomes S_1, the other focal surface of the congruence of tangent lines.

§ 8. *Congruences* Γ_N.

For congruences Γ_N, $c_1=0$, while c_2 and c_3 are any constants such that $c_2^2+c_3^2=1$. Writing $c_2=\cos\theta$, $c_3=\sin\theta$, we find that for a congruence Γ_N equations (27) become

$$\left.\begin{aligned}\rho_{01}+\rho_{02}&=\frac{L_3+\dfrac{q}{\tau}+\dfrac{\cos\theta}{\rho}(r\sin\theta-t\cos\theta)}{\dfrac{1}{\tau}(A_2\cos\theta+A_3\sin\theta)+\dfrac{L_3}{\rho}\cos\theta},\\ \rho_{01}\rho_{02}&=\frac{r\sin\theta-t\cos\theta}{\dfrac{1}{\tau}(A_2\cos\theta+A_3\sin\theta)+\dfrac{L_3}{\rho}\cos\theta}.\end{aligned}\right\} \tag{33}$$

---

* Eisenhart, p. 403.

In order that S be the middle surface* of the congruence, it is necessary and sufficient† that

$$L_3+\frac{q}{\tau}+\frac{\cos\theta}{\rho}(r\sin\theta-t\cos\theta)=0. \tag{34}$$

Since the angle θ is constant, it is clear that, in general, there are no congruences Γ_N for which (34) is satisfied. If there are two such congruences, and θ_1 and θ_2 are the corresponding values of θ, we have

$$r(\sin\theta_1\cos\theta_1-\sin\theta_2\cos\theta_2)=t(\cos^2\theta_1-\cos^2\theta_2),$$

whence

$$\tan(\theta_1+\theta_2)=-\frac{r}{t}=\tan\omega.$$

Therefore the angles θ_1 and θ_2 can be so chosen that $\theta_1+\theta_2=\omega$, and hence the number of such congruences can not exceed two. It can be equal to two only in case ω is constant.

If ω is constant, the normals to the surface form a congruence Γ_N. In order that S be the middle surface of this congruence, it is necessary and sufficient that S be minimal.‡ We conclude that, when S is minimal and ω is constant, then S is the middle surface of the congruence of principal normals. For, in view of the preceding paragraph, if (34) is satisfied by $\theta=\omega$, it is also satisfied by $\theta=0$. This result is a special case of a theorem which we shall now prove.

When $q=0$, and (34) is satisfied for $\theta=0$, we find for the mean curvature of S the expression

$$K_m=\frac{ED''+GD-2FD'}{EG-F^2}=\frac{\frac{\partial\omega}{\partial v}}{\sqrt{r^2+t^2}}.$$

If, on the other hand, we assume that

$$q=\frac{\partial\omega}{\partial v}=K_m=0,$$

it is found that (34) is satisfied for $\theta=0$. We can therefore state

THEOREM 5. *For a system of curves C on a surface S, each of the following statements is a consequence of the two remaining:*

1°. *The surface S is minimal.*

2°. *The surface S is the middle surface of the congruence of principal normals.*

---

* Eisenhart, p. 399.

† We do not consider the degenerate congruences for which $\mathfrak{E}_0\mathfrak{G}_0-\mathfrak{F}_0^2$ vanishes.

‡ Eisenhart, pp. 180, 251.

3°. *Along each of the orthogonal trajectories of the curves C, the angle ω is constant.*

It is easy to show, in view of (12), that an unlimited number of systems satisfying the condition 3° exist on every surface, independently of the cases in which ω is constant all over the surface. The condition 3° is satisfied by a system of geodesics or asymptotic lines; in each case, Theorem 5 states a well known characteristic property of minimal surfaces.

In like manner we prove

THEOREM 6. *For a system of curves C on a surface S, each of the following statements is a consequence of the two remaining:*

1°. *The orthogonal trajectories of the curves C are asymptotic lines.*

2°. *The surface S is the middle surface of the congruence of binormals.*

3°. *Along each of the orthogonal trajectories of the curves C, the angle ω is constant.*

In order that S be the middle surface of both the congruence of principal normals and the congruence of binormals, we must have

$$L_3+\frac{q}{\tau}-\frac{t}{\rho}=L_3+\frac{q}{\tau}=0.$$

Then $t=0$, and from (9) it is seen that the mean curvature of S is zero.

THEOREM 7. *The only curves C which have S for the middle surface of both the congruence of principal normals and the congruence of binormals are the asymptotic lines of a minimal surface.*

COROLLARY. *Whenever S is the middle surface of two of the congruences $\mathbf{\Gamma}_t$, $\mathbf{\Gamma}_n$ and $\mathbf{\Gamma}_b$, it is of the third also.*

Meusnier's theorem* states that all the osculating circles of the curves on S tangent to the curve C at the point P lie on a sphere M of radius $|\rho/\cos\omega|$ tangent to S at P. We shall now prove the following theorem with regard to this sphere:

THEOREM 8. *If the generating lines of two congruences $\mathbf{\Gamma}_N$ are constantly orthogonal, and the curves C bisect the angles between the curves cut out on S by the developables of one of the congruences, the focal points of the other congruence are harmonic with respect to the corresponding sphere M. If the focal points of one of the congruences are harmonic with respect to the sphere M, and the generating line of this congruence is not tangent to M, the curves C bisect the angles between the curves cut out on S by the developables of the other congruence.*

* Eisenhart, p. 118.

For simplicity, we take $q=0$. The equation of the curves cut out by the developables will be identical with (25), and the angles between these curves are bisected by the parametric curves if, and only if,*

$$\mathcal{E}_0 g_0 - \mathcal{G}_0 e_0 + \mathcal{F}_0 f_0' - \mathcal{F}_0 f_0 = 0.$$

For the congruence Γ_N for which $\theta=\theta_1$, this condition takes the form

$$L_3 - \frac{\cos\theta_1}{\rho}(r\sin\theta_1 - t\cos\theta_1) = 0. \tag{35}$$

If the coordinates of the harmonic conjugate Q of P with respect to P_{01} and P_{02} are of the form

$$x + hX_0,$$

we find that

$$h = \frac{2\rho_{01}\rho_{02}}{\rho_{01}+\rho_{02}}. \tag{36}$$

Then, for the congruence Γ_N such that $\theta=\theta_2$, we have, by (33),

$$h\left[L_3 + \frac{\cos\theta_2}{\rho}(r\sin\theta_2 - t\cos\theta_2)\right] = 2(r\sin\theta_2 - t\cos\theta_2). \tag{37}$$

Since the sphere M is of radius $|\rho/\cos\omega|$, the point Q will lie on M if

$$h = \frac{2\rho}{\cos\omega}\cos(\omega-\theta_2) = -\frac{2\rho}{t}(r\sin\theta_2 - t\cos\theta_2). \tag{38}$$

When (38) is satisfied, and $h\neq 0$, (37) becomes

$$L_3 + \frac{\sin\theta_2}{\rho}(r\cos\theta_2 + t\sin\theta_2) = 0. \tag{39}$$

If $h=0$, the generating line of the congruence is tangent to the sphere M. Conversely, if (37) and (39) are satisfied, (38) follows. To complete the proof of Theorem 8, it is sufficient to note that (35) and (39) are equivalent if $\theta_1 = \theta_2 \pm \frac{\pi}{2}$.

COROLLARY. *If the curves C bisect the angles between the curves cut out on S by the developables of the congruence of binormals, the focal points of the congruence of principal normals are harmonic conjugates with respect to the osculating circle of the curve C, and conversely.*

For the congruence Γ_n, (36) becomes

$$h = \frac{2t}{\dfrac{t}{\rho} - \left(L_3 + \dfrac{q}{\tau}\right)}; \tag{40}$$

---

* Eisenhart, p. 76.

and for the congruence Γ_b, (34) reduces to

$$L_3+\frac{q}{\tau}=0. \tag{41}$$

Then $h=2\rho$ if, and only if,* (41) is satisfied. Hence we have

THEOREM 9. *In order that S be the middle surface of the congruence of binormals, it is necessary and sufficient that the focal points of the congruence of principal normals be harmonic conjugates with respect to the osculating circles of the curves C.*

Combining Theorem 8, corollary, and Theorem 9, we obtain

THEOREM 10. *When, and only when, the surface S is the middle surface of the congruence of binormals, do the curves C bisect the angles between the curves cut out on S by the developables of the congruence.*

Interpreted for asymptotic lines, this theorem states the well known fact that the surface is minimal if, and only if, the asymptotic lines form an orthogonal system.

If we put $q=0$, and require that $h=\rho$, it follows from (40) that

$$L_3+\frac{t}{\rho}=0. \tag{42}$$

Since (35) reduces to (42) when $\theta_1=0$, we have

THEOREM 11. *In order that the center of the osculating circle be the harmonic conjugate of P with respect to the focal points of the congruence of principal normals, it is necessary and sufficient that the curves C bisect the angles between the curves cut out on S by the developables of the congruence.*

If the curves C are geodesics, the curves cut out on S by the developables of the congruence of principal normals are the lines of curvature, so that we have the

COROLLARY. *In order that a system of geodesics be one system of the mean orthogonal lines, it is necessary and sufficient that the center of the osculating circle be the harmonic conjugate of P with respect to the surfaces of center of S.*†

This corollary is also a direct consequence of the fact that the normal curvature of the surface in the direction of the mean orthogonal lines is equal to one-half the mean curvature of the surface.

For the congruence Γ_b, (36) becomes, if we again take $q=0$,

$$h=\frac{2r}{L_3},$$

---

* Exception must be made of the case $t=0$, but Theorem 9 is easily verified directly in that case.

† Eisenhart, p. 179.

and for the congruence Γ_n (34) reduces to

$$L_3 - \frac{t}{\rho} = 0. \tag{43}$$

When (43) is satisfied, it is found that

$$h = -2\rho \tan \omega. \tag{44}$$

Conversely, if (44) is satisfied and $r \neq 0$, then (43) follows. If we denote by $\bar{M}$ the sphere obtained by reflecting the sphere M of Meusnier in the tangent plane of the surface, it is readily proved that (44) may be interpreted as follows:

THEOREM 12. *If S is the middle surface of the congruence of principal normals, the focal points of the congruence of binormals are harmonic conjugates with respect to the sphere $\bar{M}$. If the focal points of the congruence of binormals are harmonic conjugates with respect to the sphere $\bar{M}$, and the curves C are not geodesics, S is the middle surface of the congruence of principal normals.*

§ 9. *The Congruence of Polar Lines.*

If we take $\xi_1 = \xi_3 = c_1 = c_2 = 0$, $\xi_2 = \rho$, $c_3 = 1$, we obtain the congruence of polar lines of the curves C. If we assume that $q = 0$, we find that

$$\rho_{01} + \rho_{02} = -\frac{\dfrac{\partial \rho}{\partial u} A_3 + \dfrac{p\rho}{\tau} A_2}{\dfrac{p}{\tau} A_3}, \quad \rho_{01}\rho_{02} = \frac{A_2 \rho \dfrac{\partial \rho}{\partial u}}{\dfrac{p}{\tau} A_3}, \tag{45}$$

whence

$$\rho_{01} = -\frac{\tau}{p}\frac{\partial \rho}{\partial u}, \quad \rho_{02} = -\frac{A_2 \rho}{A_3}. \tag{46}$$

Thus the focal point P_{01} is the center of the osculating sphere of the curve, and it is easily proved that P_{02} is the point of contact of the normal plane of the curve C with the surface of which the double infinity of normal planes of the curves C are the tangent planes.

If the curves C are lines of curvature, we have $A_3 \sin \omega + A_2 \cos \omega = 0$, whence $\rho_{02} = \rho \tan \omega$. Therefore the point P_{02} is now the intersection of the polar line with the normal to the surface. Conversely, if P_{02} is this point, it follows from (46) that $A_3 \sin \omega + A \cos \omega = 0$.

THEOREM 13. *It is a characteristic property of the lines of curvature that the point of meeting of the normal to the surface and the corresponding polar line is a focal point of the congruence of polar lines.*

This theorem may be proved directly by purely geometric considerations. If the curves C are lines of curvature, so are their orthogonal trajectories C_1. Hence, the normal planes of the curves C are tangent to the developable surfaces formed by the normals to the surface S along the curves C_1. The point of contact Q of the normal plane with its envelope therefore lies on the normal to the surface. On the other hand, the polar lines are the characteristics of the normal planes of a curve C; hence, Q lies also on the polar line.

III. Congruences of Circles.

§ 10. *Cyclic Systems of Osculating Circles.*

The osculating circles of a one-parameter family of curves C, other than circles, form a congruence. We inquire under what conditions this congruence is a *cyclic system*,* that is, a congruence of circles which admit a one-parameter family of orthogonal surfaces. We consider first, however, a slightly more general problem.

At each point P, draw in the osculating plane a circle k tangent to the curve C at P. If $R(u, v)$ is the radius of this circle, M its center, Q any point on the circle, and θ the angle which the radius MQ makes with the tangent to the curve, the coordinates of Q and the direction-cosines of the tangent to the circle k at Q are of the form

$$\bar{x}=x+\alpha R\cos\theta+lR(1+\sin\theta), \tag{47}$$

and

$$\bar{X}=\alpha\sin\theta-l\cos\theta. \tag{48}$$

In order that the surface $\bar{S}$, locus of Q, be normal to the circles k, it is necessary that $\Sigma\bar{X}d\bar{x}=0$. This condition can be written

$$Rd\theta+Adu+Bdv=0, \tag{49}$$

where

$$A=\cos\theta\frac{\partial R}{\partial u}+p\left(\frac{R}{\rho}-1\right)\sin\theta+\frac{pR}{\rho},$$

$$B=\cos\theta\frac{\partial R}{\partial v}+(A_2R-q)\sin\theta+r\cos\theta+A_2R.$$

The condition that (49) admit a solution involving a parameter is that

$$R\left(\frac{\partial A}{\partial v}-\frac{\partial B}{\partial u}\right)+A\left(\frac{\partial B}{\partial \theta}-\frac{\partial R}{\partial v}\right)+B\left(\frac{\partial R}{\partial u}-\frac{\partial A}{\partial \theta}\right) \tag{50}$$

be identically zero. The expression (50) reduces to the form

$$\Phi_1\sin\theta+\Phi_2\cos\theta+\Phi_3,$$

* Eisenhart, p. 426.

where the Φ's are functions of u and v. It is therefore identically zero if and only if

$$\Phi_1=\Phi_2=\Phi_3=0.$$

We thus obtain the three conditions

$$\left.\begin{aligned} r\left(1-\frac{R}{\rho}\right)=0, \\ (q-A_2R)\frac{\partial R}{\partial u}+p\left(\frac{R}{\rho}-1\right)\frac{\partial R}{\partial v}-\frac{p}{\tau}R^2A_3=0, \\ r\frac{\partial R}{\partial u}+p\frac{t}{\tau}R=0. \end{aligned}\right\} \tag{51}$$

Therefore, either $r=0$ or $R=\rho$. In the first case, the other two conditions reduce, in view of (1) and (2), to

$$\frac{\partial R}{\partial v}=\frac{1}{\tau}=0, \tag{52}$$

provided we take $q=0$, as we may without loss of generality. Hence, the curves C are the plane geodesics of a surface of Monge;* the only restriction on R is that it be constant along the orthogonal trajectories of the curves C. This result is included in a more general one, of which we give a geometric proof.

Every surface of Monge can be generated by a plane curve whose plane rolls, without slipping, over a developable surface.† The successive positions of the curve form the system of plane geodesics. Any single infinity of circles drawn in the plane of the curve will generate a congruence. Since the circles in the plane admit a one-parameter family of orthogonal curves, the congruence of circles admits as orthogonal trajectories the one-parameter family of surfaces generated by these curves, and is therefore a cyclic system. The normal surfaces are clearly also surfaces of Monge.

Conversely, Ribaucour‡ has shown that all cyclic systems of circles whose planes envelop a developable surface can be obtained in the manner just indicated.

If we do not have $r=0$, we must take $R=\rho$. If we again put $q=0$, the remaining conditions reduce to

$$\frac{\partial\omega}{\partial u}-\frac{p}{\tau}=0, \quad \frac{\partial}{\partial u}\left(\frac{\sin\omega}{\rho}\right)=0, \tag{53}$$

---

* G. Monge, "Application de l'Analyse à la Géométrie, § 24, Paris (1849).

† Eisenhart, p. 306.

‡ A. Ribaucour, "Mémoire sur la théorie générale des surfaces courbes," *Journal de Mathématiques*, Ser. 4, Vol. VII (1891), p. 264.

The curves C must therefore be lines of curvature* of constant geodesic curvature.† Since the plane geodesics of a surface of Monge are of this type, we are able to state

THEOREM 14. *If the curves C are not plane geodesics, the circles k do not form a cyclic system unless they are the osculating circles. The necessary and sufficient condition that the osculating circles form a cyclic system is that the curves be lines of curvature of constant geodesic curvature.*

Lines of curvature of constant geodesic curvature are, except in the case of plane geodesics already discussed, necessarily spherical curves, and lie on spheres which meet the surface orthogonally.‡ Conversely, curves on a surface, which lie also on spheres orthogonal to the surface, are lines of curvature of constant geodesic curvature.

It is at once evident that the surfaces obtained by subjecting surfaces of Monge to an inversion will have a system of spherical lines of curvature of constant geodesic curvature. In this way it is possible to obtain all such surfaces for which the corresponding spheres have a point in common. We shall now show how those not having this property can then be obtained.

All surfaces Σ with a system of spherical lines of curvature fall into two classes:§

1°. The surfaces (Σ_1) obtained by transforming by inversion all surfaces with a system of plane lines of curvature.

2°. The surfaces (Σ_2) obtained by subjecting the surfaces Σ_1 to the Combescure transformation for surfaces Σ.

The Combescure transformation is the following. The coordinates of the centers of the one-parameter family of spheres on which lie the curves $v=$const. of a surface Σ are given by three functions of v alone

$$V_1,\ V_2,\ V_3.$$

If, as usual, X, Y, Z are the direction-cosines of the normal to the surface, and α_1, β_1, γ_1 are the direction-cosines of the tangent to the curve $u=$const., and if we suppose the parametric system orthogonal, we have

$$x=\xi X+\eta\alpha_1+V_1, \tag{54}$$

and similar equations for y and z, where ξ and η are functions of v alone. If now we have any five other functions of v satisfying the equations

* See § 2.

† Eisenhart, p. 132.

‡ Darboux, "Leçons sur la théorie générale des surfaces," Vol. III, p. 121, Paris (1896).

§ Bianchi, "Lezioni di Geometria Differenziale," Vol. II, p. 305, Pisa (1903).

$$\frac{\bar{\xi}'}{\xi'}=\frac{\bar{\eta}}{\eta}=\frac{\overline{V}_1'}{V_1'}=\frac{\overline{V}_2'}{V_2'}=\frac{\overline{V}_3'}{V_3'}, \tag{55}$$

where the prime denotes differentiation with respect to v, the point whose coordinates are of the form

$$\bar{x}=\bar{\xi}X+\bar{\eta}\alpha_1+\overline{V}_1 \tag{56}$$

describes a surface $\overline{\Sigma}$, which has the same spherical representation of its lines of curvature as Σ, and of which the curves $v=$const. are spherical and lie on spheres with centers given by

$$\overline{V}_1,\ \overline{V}_2,\ \overline{V}_3.$$

When the spherical lines of curvature of Σ have constant geodesic curvature, $\xi=0$, and conversely. For then the centers of the spheres lie on the tangents to the curves $u=$const. When $\xi=0$, it does not necessarily follow that $\bar{\xi}=0$, but, if we define a *modified Combescure transformation* by requiring that $\bar{\xi}=0$ when $\xi=0$, we can prove

THEOREM 15. *All surfaces S with a system of lines of curvature of constant geodesic curvature consist of*

1°. *The surfaces (S_1) obtained by transforming by inversion all surfaces of Monge.*

2°. *The surfaces (S_2) obtained by subjecting the surfaces S_1 to the modified Combescure transformation.*

To prove this theorem, it must be shown that, if $\xi=\bar{\xi}=0$, and

$$\frac{\bar{\eta}}{\eta}=\frac{\overline{V}_1'}{V_1'}=\frac{\overline{V}_2'}{V_2'}=\frac{\overline{V}_3'}{V_3'}=\varkappa, \tag{57}$$

it is always possible so to determine $\varkappa$ that, after the transformation, the spheres to which the surface is orthogonal have a point in common. This fact is readily established by a suitable modification of the discussion given by Darboux* to prove the corresponding fact for the general Combescure transformation.

It is at once evident that the surfaces normal to the osculating circles of the spherical lines of curvature of a surface of the class (S_1) belong themselves to that class. That the corresponding theorem for surfaces of the class (S_2) is true is easily proved, in view of the theorem that, when a surface cuts a sphere orthogonally, the intersection is a line of curvature of the surface.

§ 11. *Cyclic Systems of Circles on a One-Parameter Family of Spheres.*

The cyclic systems discussed in the preceding section had the property that the circles were on a one-parameter family of planes or of spheres. We

* Darboux, "Leçons," Vol. IV, p. 249.

showed how to obtain all cyclic systems for which the circles were on a single infinity of planes. In this section, we determine all cyclic systems for which the circles are on a single infinity of spheres. We must first demonstrate some properties of the modified Combescure transformation.

Consider any one-parameter family of spheres admitting a family of orthogonal surfaces S. These surfaces are necessarily of class (S_1) or (S_2). Let S_0 be one of the surfaces S, and subject it to the modified Combescure transformation given by (57). If

$$x_0=\eta\alpha_1+V_1, \quad y_0=\eta\beta_1+V_2, \quad z_0=\eta\gamma_1+V_3,$$

then, by (56),

$$\left.\begin{aligned}\bar{x}_0&=\bar{\eta}\alpha_1+\overline{V}_1\\&=\varkappa\eta\alpha_1+\int\varkappa V_1'dv\\&=\varkappa(\eta\alpha_1+V_1)-\int\varkappa' V_1dv\\&=\varkappa x_0-\int\varkappa' V_1dv,\end{aligned}\right\} \tag{58}$$

and, similarly,

$$\left.\begin{aligned}\bar{y}_0&=\varkappa y_0-\int\varkappa' V_2dv,\\ \bar{z}_0&=\varkappa z_0-\int\varkappa' V_3dv.\end{aligned}\right\} \tag{59}$$

Equations (58) and (59) define, for each value of v, a linear transformation with constant coefficients. This transformation is independent of the particular surface S which is transformed. It depends only on the quantities $\varkappa$, V_1, V_2, V_3, and may be regarded as a uniform magnification leaving the origin invariant, followed by a translation. Since the points (of the surfaces S) which correspond to a given value of v all lie on the same sphere, we observe that the transformation of the surfaces S sets up a one-to-one correspondence between the points of each of the old spheres and the points of the corresponding new sphere.* In view of these remarks, we are able to state

THEOREM 16. *A given modified Combescure transformation carries all the surfaces orthogonal to a given one-parameter family of spheres over into surfaces orthogonal to a common set of spheres. The points on each of the original spheres are carried over into the points of the corresponding new sphere by a linear transformation, which varies with the sphere, but has, in each case, the properties:*

1°. *Circles are carried into circles.*

2°. *Angles are preserved.*

If the circles of a cyclic system lie on a one-parameter family of spheres through a common point, the cyclic system can be obtained by inversion from

* It is immaterial whether all points of the spheres are considered, or only those which lie on one or more of the orthogonal surfaces.

one in which the circles lie on a one-parameter family of planes. If the spheres do not pass through a common point, it is evident that the orthogonal surfaces are of the type S_2. As was remarked in § 10, there exist modified transformations of Combescure which carry these orthogonal surfaces of type S_2 over into surfaces of type S_1 orthogonal to a family of spheres through a common point. We have just shown that the same transformation may be used for all the orthogonal surfaces, and, by Theorem 16, the congruence of circles orthogonal to the surfaces S_2 is carried over into a congruence of circles orthogonal to the surfaces S_1. We have thus proved:

THEOREM 17. *Every cyclic system such that the circles lie on a single infinity of spheres can be obtained by:*

1°. *Subjecting to inversion the cyclic systems of circles whose planes envelop a developable surface.*

2°. *Subjecting to the modified transformation of Combescure the orthogonal surfaces of the cyclic systems given by* 1°.

The circles of one of these cyclic systems may be regarded as forming a one-parameter family of tubular surfaces, which are easily proved to be of type (S_1) or (S_2). Hence, 2° may be replaced by

2°. *Subjecting to the modified transformation of Combescure the tubular surfaces generated by the circles of the cyclic systems given by* 1°, *as a single sphere is made to take on in succession the positions of the spheres of the given family.*

IV. O-SURFACES.

§ 12. *Definition. Differential Equation.*

The characteristic property of an orthogonal system is that, at their points of meeting, the curves have orthogonal tangents. If we also require the principal normals to be orthogonal, the system must consist of asymptotic lines and their orthogonal trajectories; if we, on the other hand, add the requirement that the binormals be orthogonal, the system must consist of geodesics and their orthogonal trajectories.* In order to obtain all three properties simultaneously, it is necessary to consider geodesics whose geodesic parallels are asymptotic lines. Such a system will be called an *O-system*, a surface with one such system an *O-surface*, and a surface with two such systems a *double O-surface*.

The consideration of O-surfaces is also suggested by the notions of the second part of this paper. For the principal normals of a system of asymptotic

* Cf. §§ 2, 4.

lines form a normal congruence only in case the asymptotic lines are geodesic parallels, and the congruence of binormals of a geodesic system on a surface S has S for its middle surface only in case the orthogonal trajectories of the geodesics are asymptotic lines.

In order to determine whether any O-systems can exist, we must seek solutions of the fundamental equations (1) such that either

$$q=r=t_1=0,$$

or

$$q=r_1=t=0.$$

Making the first choice, we have also

$$\frac{\partial p}{\partial v}=0,$$

and can put $p=1$.* The equations (1) and the condition $t_1=0$ now take the form

$$\left.\begin{aligned} \frac{\partial^2 t}{\partial u^2}-\frac{t}{\tau^2}&=0,\\ \frac{1}{\rho}\frac{\partial t}{\partial u}+\frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)&=0,\\ \frac{2}{\tau}\frac{\partial t}{\partial u}+t\frac{\partial}{\partial u}\left(\frac{1}{\tau}\right)-\frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)&=0. \end{aligned}\right\} \tag{60}$$

Eliminating ρ and τ, it is found that t must satisfy the partial differential equation of the fourth order

$$\frac{\partial}{\partial v}\left[\frac{1}{\frac{\partial t}{\partial u}}\frac{\partial}{\partial v}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}\right]+t\frac{\partial}{\partial u}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}+2\frac{\partial t}{\partial u}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}=0, \tag{61}$$

and that ρ and τ are then given by

$$\frac{1}{\rho}=-\frac{1}{\frac{\partial t}{\partial u}}\frac{\partial}{\partial v}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}, \tag{62}$$

and

$$\frac{1}{\tau}=\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}. \tag{63}$$

Thus far we have always excluded the possibility that $1/\rho=0$, since the equations (1) then become meaningless. However, a special investigation of

* Cf. § 2.

this case proves that equations (60)–(63) are still valid when $1/\rho=0$. They are then easily integrated, and lead, in view of the theorem of Catalan,* to the right helicoid. For the curves $v=\text{const.}$ are now rulings and the surface is therefore minimal.

If $1/\tau=0$, $1/\rho \neq 0$, equations (60) become

$$\frac{\partial^2 t}{\partial u^2}=0, \quad \frac{\partial t}{\partial u}=0, \quad \frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)=0.$$

We can then take $t=1$, and ρ an arbitrary function of u. Referring to (2) and (3), we find that λ, μ and ν are then constant; or, from (20) and (22), it follows that $1/\rho_1=0$. Either of these facts shows that the O-surfaces which correspond to the assumption $1/\tau=0$, $1/\rho\neq 0$ are all cylinders. Moreover, every cylinder is clearly an O-surface. If $1/\tau=1/\rho=0$, it is found that the direction-cosines of the normal to the surface are constant; hence the surface is a plane.

§ 13. *Fundamental Quantities. Special Parametric Systems.*

We find that

$$\left.\begin{array}{llll} E=1, & F=0, & G=t^2, & H=\varepsilon t, \\ D=-\dfrac{\varepsilon}{\rho}, & D'=-\dfrac{\varepsilon t}{\tau}, & D''=0, & \end{array}\right\} \tag{64}$$

where $\varepsilon=-\cos\omega$ and is therefore either 1 or -1. Then the total and mean curvatures are given by

$$K=-\frac{1}{\tau^2}, \quad K_m=-\frac{\varepsilon}{\rho}. \tag{65}$$

These values of K and K_m could have been determined without reference to (64). For Enneper's theorem† states that

$$K=-\frac{1}{\tau_1^2},$$

where τ_1 is the radius of torsion of the asymptotic line $u=\text{const.}$, and, from the theorem that the geodesic torsions in two orthogonal directions differ only in sign,‡ follows

$$\frac{1}{\tau_1}=-\frac{1}{\tau}.$$

Moreover, it is well known that the mean curvature of any surface is equal to the normal curvature of the orthogonal trajectories of the asymptotic lines.

---

* Eisenhart, p. 148. † Eisenhart, p. 140. ‡ Eisenhart, p. 139.

Conversely, it is readily proved that, if any surface has a system of geodesics such that either

$$K = -\frac{1}{\tau^2}$$

or

$$K_m = \frac{\cos\omega}{\rho},$$

the surface is necessarily an O-surface.

From (65) and the results of the last section, it follows that the only developable O-surfaces are the plane and the cylinders; the only minimal O-surfaces are the right helicoids.

We seek now conditions under which a surface referred to its asymptotic lines or its lines of curvature is an O-surface. If S is referred to its asymptotic lines, the condition that it be an O-surface, with the curves $v=\text{const.}$ geodesic parallels, is that the curves defined by

$$p\,du + q\,dv = 0$$

be geodesics. This condition is most easily deduced from the considerations of § 6. We found there that the necessary and sufficient condition that the congruence of principal normals of the curves $v=\text{const.}$ form a normal congruence is that r be a function of v alone. We also found that

$$r = \frac{H}{\sqrt{E}}\sin\omega,$$

whence, in this case,

$$r = \pm\frac{H}{\sqrt{E}}.$$

We can therefore state

THEOREM 18. *The necessary and sufficient condition that a surface referred to its asymptotic lines be an O-surface is that one of the two conditions*

$$\frac{\partial}{\partial u}\left(\frac{H^2}{E}\right)=0, \quad \frac{\partial}{\partial v}\left(\frac{H^2}{G}\right)=0 \tag{66}$$

be satisfied. If both conditions are satisfied, the surface is a double O-surface. When the first of (66) *is satisfied, the parameter v can be so chosen that the linear element is*

$$E\,du^2 + 2\sqrt{E(G-1)}\,du\,dv + G\,dv^2; \tag{67}$$

when both conditions are satisfied, the linear element can take the form

$$E\,du^2 + 2\sqrt{E(E-1)}\,du\,dv + E\,dv^2. \tag{68}$$

Moreover, these linear elements are characteristic of O surfaces referred to their asymptotic lines.

If we transform the parameters of (68) by putting

$$u=u_1+v_1, \quad v=u_1-v_1,$$

the new parametric curves are the lines of curvature and the first and second fundamental forms become

$$2(E+\sqrt{E(E-1)})\,du_1^2+2(E-\sqrt{E(E-1)})\,dv_1^2=E_1du_1^2+G_1dv_1^2,$$

and

$$2D'du_1^2-2D'dv_1^2=D_1du_1^2+D_1''dv_1^2.$$

Thus

$$\frac{1}{E_1}+\frac{1}{G_1}=1, \quad D_1+D_1''=0, \tag{69}$$

Conversely, if the lines of curvature are parametric and equations (69) are satisfied, by the substitution

$$u_1=\frac{u+v}{2}, \quad v_1=\frac{u-v}{2},$$

the asymptotic lines are made parametric, and the linear element assumes the form (68). Thus we have

THEOREM 19. *In order that a surface referred to its lines of curvature be a double O-surface, it is necessary and sufficient that the parameters can be so chosen that*

$$\frac{1}{E}+\frac{1}{G}=1, \quad D+D''=0.$$

The lines of curvature of a double O-surface form an isothermal-conjugate system and are geodesic ellipses and hyperbolas.†*

§ 14. *O-Surfaces which are Surfaces of Weingarten. Double O-Surfaces.*

In view of (65), the necessary and sufficient condition that an O-surface be a surface of Weingarten‡ is that ρ be a function of τ. If, for brevity, we put

$$Q=\frac{\dfrac{\partial^2 t}{\partial u^2}}{t},$$

equations (61)–(63) become

$$\frac{\partial^2 Q}{\partial v^2}-\frac{\dfrac{\partial^2 t}{\partial u\partial v}}{\dfrac{\partial t}{\partial u}}\frac{\partial Q}{\partial v}+t\frac{\partial t}{\partial u}\frac{\partial Q}{\partial u}-\frac{1}{2Q}\left(\frac{\partial Q}{\partial v}\right)^2+4Q\left(\frac{\partial t}{\partial u}\right)^2=0, \tag{70}$$

$$\frac{1}{\tau}=\sqrt{Q}, \tag{71}$$

* Eisenhart, p. 198. † Eisenhart, p. 213. ‡ Eisenhart, p. 291.

$$\frac{1}{\rho}=-\frac{1}{2\sqrt{Q}}\frac{\frac{\partial Q}{\partial v}}{\frac{\partial t}{\partial u}}. \tag{72}$$

The condition that ρ be a function of τ then becomes

$$\frac{\partial^2 Q}{\partial u\partial v}-tQ\frac{\frac{\partial Q}{\partial v}}{\frac{\partial t}{\partial u}}+t\frac{\frac{\partial t}{\partial u}\left(\frac{\partial Q}{\partial u}\right)^2}{\frac{\partial Q}{\partial v}}-\frac{\frac{\partial Q}{\partial u}\,\frac{\partial Q}{\partial v}}{2Q}+4\frac{Q\left(\frac{\partial t}{\partial u}\right)^2\frac{\partial Q}{\partial u}}{\frac{\partial Q}{\partial v}}=0. \tag{73}$$

In order to determine all O-surfaces which are W-surfaces, it would be necessary to find all common solutions of (70) and (73). That they are consistent is seen by supposing that

$$\frac{\partial t}{\partial u}=\frac{\partial t}{\partial v}, \tag{74}$$

for they then become identical.

In view of a well known theorem,* every solution of (70) which satisfies (74) corresponds to a surface which is either a helicoid or a surface of revolution, since t, ρ and τ are then functions of $u+v$. When (74) is satisfied, (70) reduces to the ordinary differential equation of the fourth order

$$t''''-\frac{t'\,t'''}{t}-\frac{t''\,t'''}{t'}-\frac{1}{2}\frac{t'''^2}{t''}+t\,t'\,t'''+\frac{3}{2}\frac{t'^2\,t''}{t^2}+3t'^2\,t''=0. \tag{75}$$

In order that an O-surface be double,

$$\phi=\frac{2\rho}{t\tau}=-\frac{4Q}{t}\frac{\frac{\partial t}{\partial u}}{\frac{\partial Q}{\partial v}}$$

must satisfy the equation of geodesics

$$\frac{\partial\phi}{\partial u}+\phi\frac{\partial\phi}{\partial v}+t\frac{\partial t}{\partial u}\phi^3+\frac{\frac{\partial t}{\partial v}}{t}\phi^2+2\frac{\frac{\partial t}{\partial u}}{t}\phi=0, \tag{76}$$

so that

$$\frac{\partial^2 Q}{\partial u\partial v}-\frac{tQ\frac{\partial Q}{\partial v}}{\frac{\partial t}{\partial u}}+\frac{\frac{\partial t}{\partial u}\,\frac{\partial Q}{\partial v}}{t}-\frac{\frac{\partial Q}{\partial u}\,\frac{\partial Q}{\partial v}}{Q}+\frac{4Q\left(\frac{\partial t}{\partial u}\right)^2\frac{\partial Q}{\partial u}}{\frac{\partial Q}{\partial v}}=0. \tag{77}$$

As before, we merely show that (70) and (77) are consistent and do not attempt to find all their common solutions. If (74) is satisfied, (70) and (77) are consistent only if

---

* A. Enneper, *Göttingen Nachrichten*, 1870, p. 335.

$$\frac{\left(\frac{\partial Q}{\partial u}\right)^2}{2Q}-\frac{\frac{\partial t}{\partial u}\frac{\partial Q}{\partial u}}{t}+t\frac{\partial t}{\partial u}\frac{\partial Q}{\partial u}=0.$$

Disregarding the trivial case in which

$$\frac{\partial Q}{\partial u}=0,$$

we must take

$$\frac{\partial Q}{\partial u}=2Q\left(\frac{\frac{\partial t}{\partial u}}{t}-t\frac{\partial t}{\partial u}\right). \tag{78}$$

It is found that, when (74) is satisfied, every solution of (78) satisfies both (70) and (77). Hence the O-surfaces which correspond to solutions of the ordinary differential equation of the third order

$$t'''-3\frac{t'\,t''}{t}+2t\,t'\,t''=0 \tag{79}$$

are double O-surfaces and at the same time are W-surfaces. The solution of (79) can be reduced to a single quadrature, and is given by

$$\int\frac{dt}{\sqrt{c_2-c_1e^{-t^2}(1+t^2)}}=u+v+c_3.$$

§ 15. *Two Characteristic Properties of O-Surfaces.*

In case the curves C are geodesics, Theorem 9 affords the following characterization of O-surfaces, which is also a direct consequence of (65):

THEOREM 20. *In order that a surface S be an O-surface, it is necessary and sufficient that there be on it a system of geodesics, with respect to the osculating circles of which the corresponding points of the two surfaces of center of S are harmonic conjugates.*

The considerations of § 7 lead to another characteristic property of O-surfaces. When $t=0$, (32) becomes

$$\bar{x}=x+\frac{V'\tau}{r}\alpha-V\lambda.$$

If, furthermore, S is an O-surface with the curves $v=$const. geodesic parallels, we have by § 13

$$\frac{\partial r}{\partial u}=0,$$

whence, taking $r=1$, $V=v$, the point whose coordinates are of the form

$$\bar{x}=x+\tau\alpha-v\lambda$$

generates a surface with the same spherical representation of the parametric lines as S.

Conversely, if S is any surface with the curves $v=$const. asymptotic lines, and the congruence Γ given by

$$x_0=x+\tau\alpha, \quad X_0=\lambda$$

is normal, we must have

$$\frac{\partial}{\partial u}(\tau A_3)=-\frac{\partial r}{\partial u}=0,$$

so that the surface S is an O-surface. Thence follows

THEOREM 21. *Let S be any surface, and, on each asymptotic line of one system, mark, on the positive* half-tangent, the point P_1 whose distance from the point P of contact is equal to the radius of torsion of the asymptotic line, and hence equal, to within the sign, to*

$$\sqrt{-\frac{1}{K}},$$

where K is the total curvature of S. Then the necessary and sufficient condition that S be an O-surface, with the asymptotic lines in question as geodesic parallels, is that the parallel, through P_1, to the normal to the surface shall generate a normal congruence.

PRINCETON UNIVERSITY, *June*, 1914.

* The negative would do equally well.